Conservation equations for multispecies non-equilibrium arc plasmas

M. S. Benilov, Dec. 4, 2019

This is supplementary material to the paper [1]. Before reading, please check the page https://fisica.uma.pt/public-domain/supplements-to-papers/ for eventual updates.

The paper [1] is concerned with numerical investigation of AC arc ignition on cold electrodes in atmospheric-pressure argon. The numerical model is based on a system of differential equations, comprising conservation and transport equations for all plasma species, the electron and heavy-particle energy equations, and the Poisson equation. There is a significant amount of high-quality research dedicated to the modelling of non-equilibrium arc plasmas (e.g., reviews [2–4] and references therein). However, one finds, somewhat surprisingly, that there is no universal agreement in the literature on the proper form of non-stationary term in the electron energy equation; in particular, this term in the Plasma module of COMSOL Multiphysics® [5] appears to be incorrect.

In such a situation, it seems methodologically correct to re-derive the electron energy equation from the first principles, without relying on previous work. This is the purpose of this text. The system of conservation equations for multispecies non-equilibrium arc plasmas is obtained by integrating the species Boltzmann equations. The procedure is similar to that of sections 4.1 and 14.4 of the book [6], with minor changes. However, the results are not quite the same.

1 The Boltzmann and conservation equations for species

Let us consider a multicomponent ideal gas mixture. The Boltzmann equation governing the distribution function $f_\alpha = f_\alpha\left(\mathbf{r}, \mathbf{c}_\alpha, t\right)$ of a species α in the mixture is written in the conventional form

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{c}_\alpha \cdot \nabla f_\alpha + \mathbf{F}_\alpha \cdot \frac{\partial f_\alpha}{\partial \mathbf{c}_\alpha} = \sum_\beta J_{\alpha\beta} + \Gamma_\alpha, \tag{1}$$

where $\mathbf{c}_\alpha$ is the particle velocity, $\mathbf{F}_\alpha$ is a force per unit mass acting on a particle of species α (it is assumed that this force does not depend on $\mathbf{c}_\alpha$), the term $J_{\alpha\beta}$ accounts for elastic collisions with a species β, the term Γ_α accounts for production and loss of particles of the species α as a result of reactions (inelastic collisions, collisions with chemical transformations, and radiative processes), and the sum on the rhs is taken over all species of the mixture.

Equations of conservation of mass, momentum, and energy of each species may be obtained in the usual way by means of integration of the Boltzmann equation with the appropriate weights (m_α, $m_\alpha \mathbf{c}_\alpha$, and $m_\alpha c_\alpha^2/2$, respectively, m_α being the mass of a particle of the considered species) and read

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot \left[\rho_\alpha \left(\mathbf{v} + \mathbf{V}_\alpha\right)\right] = m_\alpha w_\alpha, \tag{2}$$

$$\frac{\partial}{\partial t}\left[\rho_\alpha \left(\mathbf{v} + \mathbf{V}_\alpha\right)\right] + \nabla \cdot \left[\rho_\alpha \left(\mathbf{V}_\alpha \mathbf{v} + \mathbf{v}\mathbf{V}_\alpha + \mathbf{v}\mathbf{v}\right)\right] = -\nabla \cdot \hat{\mathbf{P}}_\alpha + \rho_\alpha \mathbf{F}_\alpha - \sum_\beta \mathbf{r}_{\alpha\beta}^{(m)} + \mathbf{w}_\alpha^{(m)}, \tag{3}$$

$$
\frac{\partial}{\partial t}\left[\rho_\alpha\left(u_\alpha+\mathbf{V}_\alpha\cdot\mathbf{v}+\frac{v^2}{2}\right)\right]+\nabla\cdot\left[\rho_\alpha\left(u_\alpha+\mathbf{V}_\alpha\cdot\mathbf{v}+\frac{v^2}{2}\right)\mathbf{v}+\rho_\alpha\frac{v^2}{2}\mathbf{V}_\alpha\right]=
$$
$$
-\nabla\cdot\left(\mathbf{q}_\alpha+\hat{\mathbf{P}}_\alpha\cdot\mathbf{v}\right)+\rho_\alpha\left(\mathbf{v}+\mathbf{V}_\alpha\right)\cdot\mathbf{F}_\alpha-\sum_\beta r^{(e)}_{\alpha\beta}+w^{(e)}_\alpha. \tag{4}
$$

Here ρ_α, $\mathbf{V}_\alpha$, $\hat{\mathbf{P}}_\alpha$, u_α, and $\mathbf{q}_\alpha$ are the mass density, the diffusion velocity, the momentum-flux tensor, the thermal energy per unit mass, and the thermal energy flux density of the species α; $\mathbf{v}$ is the mean mass velocity of the mixture; $\mathbf{r}^{(m)}_{\alpha\beta}$ and $r^{(e)}_{\alpha\beta}$ are rates of loss per unit volume of, respectively, momentum and energy of the species α due to elastic collisions with a species β; w_α, $\mathbf{w}^{(m)}_\alpha$, and $w^{(e)}_\alpha$ are rates of change of the number density, momentum, and energy of the species α due to reactions.

ρ_α, $\mathbf{V}_\alpha$, $\hat{\mathbf{P}}_\alpha$, u_α, and $\mathbf{q}_\alpha$ are related to the distribution function of the species α by the conventional formulas

$$
\rho_\alpha=n_\alpha m_\alpha,\quad n_\alpha=\int f_\alpha\,d^3c_\alpha,\quad \mathbf{V}_\alpha=\frac{1}{n_\alpha}\int\mathbf{C}_\alpha f_\alpha\,d^3c_\alpha, \tag{5}
$$

$$
\hat{\mathbf{P}}_\alpha=\int m_\alpha\mathbf{C}_\alpha\mathbf{C}_\alpha f_\alpha\,d^3c_\alpha, \tag{6}
$$

$$
u_\alpha=\frac{1}{n_\alpha}\int\frac{1}{2}C_\alpha^2 f_\alpha\,d^3c_\alpha,\quad \mathbf{q}_\alpha=\int\frac{1}{2}m_\alpha C_\alpha^2\mathbf{C}_\alpha f_\alpha\,d^3c_\alpha. \tag{7}
$$

Here n_α and $\mathbf{C}_\alpha=\mathbf{c}_\alpha-\mathbf{v}$ are the number density and the peculiar velocity of the particles of species α.

One can introduce also the mean velocity of the considered species

$$
\mathbf{v}_\alpha=\frac{1}{n_\alpha}\int\mathbf{c}_\alpha f_\alpha\,d^3c_\alpha. \tag{8}
$$

The mean velocity is related to the diffusion velocity of the species as $\mathbf{v}_\alpha=\mathbf{V}_\alpha+\mathbf{v}$. Note that the peculiar velocity $\mathbf{C}_\alpha$ is defined in this work with the reference to the mean mass velocity of the mixture $\mathbf{v}$ as in [6], rather than with the reference to the mean velocity of the considered species $\mathbf{v}_\alpha$ as in [7]. This means, in particular, that quantities $\hat{\mathbf{P}}_\alpha$, u_α, and $\mathbf{q}_\alpha$ considered in this work are different from those of [7] and, consequently, equations (2)-(4) are different as well.

The mean mass velocity and the mass density of the mixture are defined as

$$
\mathbf{v}=\frac{1}{\rho}\sum_\alpha\rho_\alpha\mathbf{v}_\alpha,\quad \rho=\sum_\alpha\rho_\alpha. \tag{9}
$$

$\mathbf{r}^{(m)}_{\alpha\beta}$, $r^{(e)}_{\alpha\beta}$, w_α, $\mathbf{w}^{(m)}_\alpha$, and $w^{(e)}_\alpha$ represent the respective terms of the Boltzmann equation integrated with the appropriate weights

$$
\left\{\begin{array}{c}\mathbf{r}^{(m)}_{\alpha\beta}\\ r^{(e)}_{\alpha\beta}\end{array}\right\}=-\int\left\{\begin{array}{c}m_\alpha\mathbf{c}_\alpha\\ m_\alpha c_\alpha^2/2\end{array}\right\}J_{\alpha\beta}\,d^3c_\alpha,\quad \left\{\begin{array}{c}w_\alpha\\ \mathbf{w}^{(m)}_\alpha\\ w^{(e)}_\alpha\end{array}\right\}=\int\left\{\begin{array}{c}1\\ m_\alpha\mathbf{c}_\alpha\\ m_\alpha c_\alpha^2/2\end{array}\right\}\Gamma_\alpha\,d^3c_\alpha. \tag{10}
$$

Quantities $\mathbf{r}^{(m)}_{\alpha\beta}$ and $r^{(e)}_{\alpha\beta}$ are antisymmetric with respect to indices α and β due to conservation of momentum and energy in elastic collisions. Due to conservation of mass and momentum in reactive collisions,

$$
\sum_\alpha\left\{\begin{array}{c}m_\alpha w_\alpha\\ \mathbf{w}^{(m)}_\alpha\end{array}\right\}=\left\{\begin{array}{c}0\\ 0\end{array}\right\}. \tag{11}
$$

2 Equivalent form of species conservation equations

Equation (2) may be rewritten in the equivalent form

$$\frac{d\rho_\alpha}{dt} + \rho_\alpha \nabla \cdot \mathbf{v} + \nabla \cdot (\rho_\alpha \mathbf{V}_\alpha) = m_\alpha w_\alpha, \tag{12}$$

where $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is the material derivative.

Multiplying equation (2) by $\mathbf{v}$ and subtracting the result from equation (3), one can rewrite the latter equation in the equivalent form

$$\rho_\alpha \frac{d\mathbf{v}}{dt} + \frac{\partial}{\partial t}(\rho_\alpha \mathbf{V}_\alpha) + \rho_\alpha (\mathbf{V}_\alpha \cdot \nabla) \mathbf{v} + \nabla \cdot (\rho_\alpha \mathbf{V}_\alpha \mathbf{v}) = -\nabla \cdot \hat{\mathbf{P}}_\alpha + \rho_\alpha \mathbf{F}_\alpha - \sum_\beta \mathbf{r}_{\alpha\beta}^{(m)} + \mathbf{W}_\alpha^{(m)}, \tag{13}$$

where

$$\mathbf{W}_\alpha^{(m)} = \mathbf{w}_\alpha^{(m)} - m_\alpha \mathbf{v} w_\alpha = \int m_\alpha \mathbf{C}_\alpha \Gamma_\alpha \, d^3 c_\alpha. \tag{14}$$

Let us multiply equation (2) by $v^2/2$ and add equation (13) multiplied by $\mathbf{v}$. The resulting equation may be written as

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\rho_\alpha \frac{v^2}{2}\right) + \nabla \cdot \left[\rho_\alpha \frac{v^2}{2}(\mathbf{v} + \mathbf{V}_\alpha)\right] + \mathbf{v} \cdot \frac{\partial}{\partial t}(\rho_\alpha \mathbf{V}_\alpha) + \mathbf{v} \cdot [\nabla \cdot (\rho_\alpha \mathbf{V}_\alpha \mathbf{v})] \\ = \; &-\mathbf{v} \cdot \left(\nabla \cdot \hat{\mathbf{P}}_\alpha\right) + \rho_\alpha \mathbf{v} \cdot \mathbf{F}_\alpha - \sum_\beta \mathbf{v} \cdot \mathbf{r}_{\alpha\beta}^{(m)} + \mathbf{w}_\alpha^{(m)} \cdot \mathbf{v} - m_\alpha \frac{v^2}{2} w_\alpha. \end{aligned} \tag{15}$$

Subtracting this equation from equation (4), one can rewrite the latter equation in the equivalent form

$$\frac{d}{dt}(\rho_\alpha u_\alpha) + \rho_\alpha u_\alpha \nabla \cdot \mathbf{v} = -\nabla \cdot \mathbf{q}_\alpha - \hat{\mathbf{P}}_\alpha : (\nabla \mathbf{v}) + \rho_\alpha \mathbf{V}_\alpha \cdot \left(\mathbf{F}_\alpha - \frac{d\mathbf{v}}{dt}\right) - \sum_\beta R_{\alpha\beta}^{(e)} + W_\alpha^{(e)}. \tag{16}$$

where

$$R_{\alpha\beta}^{(e)} = r_{\alpha\beta}^{(e)} - \mathbf{v} \cdot \mathbf{r}_{\alpha\beta}^{(m)} = -\int \frac{m_\alpha C_\alpha^2}{2} J_{\alpha\beta} \, d^3 c_\alpha, \tag{17}$$

$$W_\alpha^{(e)} = w_\alpha^{(e)} - \mathbf{v} \cdot \mathbf{w}_\alpha^{(m)} + m_\alpha \frac{v^2}{2} w_\alpha = \int \frac{m_\alpha C_\alpha^2}{2} \Gamma_\alpha \, d^3 c_\alpha. \tag{18}$$

3 Conservation equations for the mixture on the whole

Let us sum each of the equations (2)-(4) over α. Taking into account that

$$\sum_\alpha \rho_\alpha \mathbf{V}_\alpha = 0, \tag{19}$$

one finds

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{20}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla \cdot \hat{\mathbf{P}} + \sum_\alpha \rho_\alpha \mathbf{F}_\alpha, \tag{21}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\rho\left(u+\frac{v^2}{2}\right)\right]+\nabla\cdot\left[\rho\left(u+\frac{v^2}{2}\right)\mathbf{v}\right]= \\
=\quad& -\nabla\cdot\left(\mathbf{q}+\hat{\mathbf{P}}\cdot\mathbf{v}\right)+\sum_\alpha \rho_\alpha\left(\mathbf{v}+\mathbf{V}_\alpha\right)\cdot\mathbf{F}_\alpha+\sum_\alpha w_\alpha^{(e)}.
\end{aligned}
\tag{22}
$$

Here $\hat{\mathbf{P}}$ is the tensor of the mixture momentum flux, u is the thermal energy per unit mass of the mixture, and $\mathbf{q}$ is the heat flux density of the mixture,

$$
\hat{\mathbf{P}}=\sum_\alpha \hat{\mathbf{P}}_\alpha,\quad u=\sum_\alpha \frac{\rho_\alpha}{\rho}u_\alpha,\quad \mathbf{q}=\sum_\alpha \mathbf{q}_\alpha. \tag{23}
$$

Equation (20) may be rewritten in the equivalent form

$$
\frac{d\rho}{dt}+\rho\nabla\cdot\mathbf{v}=0. \tag{24}
$$

Multiplying equation (20) by $\mathbf{v}$ and subtracting the result from equation (21), one obtains an equivalent form of the latter equation

$$
\rho\frac{d\mathbf{v}}{dt}=-\nabla\cdot\hat{\mathbf{P}}+\sum_\alpha \rho_\alpha\mathbf{F}_\alpha. \tag{25}
$$

4 Equivalent form of species conservation equations II

Multiplying equation (24) by ρ_α/ρ and subtracting the result from equation (12), one obtains a convenient form of the equation of conservation of number of species

$$
\rho\frac{d}{dt}\left(\frac{\rho_\alpha}{\rho}\right)+\nabla\cdot\left(\rho_a\mathbf{V}_\alpha\right)=m_\alpha w_\alpha. \tag{26}
$$

Multiplying equation (25) by ρ_α/ρ and subtracting the result from equation (13), one obtains a convenient form of the equation of conservation of momentum of species

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho_\alpha\mathbf{V}_\alpha\right)+\rho_\alpha\left(\mathbf{V}_\alpha\cdot\nabla\right)\mathbf{v}+\nabla\cdot\left(\rho_\alpha\mathbf{V}_\alpha\mathbf{v}\right) \\
=\quad& -\nabla p_\alpha+\nabla\cdot\hat{\boldsymbol{\tau}}_\alpha+\rho_\alpha\mathbf{F}_\alpha-\sum_\beta \mathbf{r}_{\alpha\beta}^{(m)}+\mathbf{W}_\alpha^{(m)}+\frac{\rho_\alpha}{\rho}\left(\nabla p-\nabla\cdot\hat{\boldsymbol{\tau}}-\sum_\beta \rho_\beta\mathbf{F}_\beta\right),
\end{aligned}
\tag{27}
$$

where p_α and $\hat{\boldsymbol{\tau}}_\alpha$ are the hydrostatic pressure and the viscous stress tensor of the species α, p and $\hat{\boldsymbol{\tau}}$ are the pressure and the viscous stress tensor of the mixture,

$$
p_\alpha=\frac{1}{3}\int m_\alpha C_\alpha^2 f_\alpha\, d^3c_\alpha,\quad \hat{\boldsymbol{\tau}}_\alpha=-\hat{\mathbf{P}}_\alpha+p_\alpha\hat{\mathbf{I}},\quad p=\sum_\alpha p_\alpha,\quad \hat{\boldsymbol{\tau}}=\sum_\alpha \hat{\boldsymbol{\tau}}_\alpha. \tag{28}
$$

Here $\hat{\mathbf{I}}$ is the identity tensor. Note that $p_\alpha=\frac{2}{3}\rho_\alpha u_\alpha$.

Multiplying equation (24) by $\rho_\alpha u_\alpha/\rho$ and subtracting the result from equation (16), one obtains a convenient form of the equation of conservation of energy of species

$$
\rho\frac{d}{dt}\left(\frac{\rho_\alpha}{\rho}u_\alpha\right)=-\nabla\cdot\mathbf{q}_\alpha-p_\alpha\nabla\cdot\mathbf{v}+\hat{\boldsymbol{\tau}}_\alpha:(\nabla\mathbf{v})+\rho_\alpha\mathbf{V}_\alpha\cdot\left(\mathbf{F}_\alpha-\frac{d\mathbf{v}}{dt}\right)-\sum_\beta R_{\alpha\beta}^{(e)}+W_\alpha^{(e)} \tag{29}
$$

or in terms of the enthalphy $h_\alpha = u_\alpha + p_\alpha/\rho_\alpha$ of the species α

$$\rho\frac{d}{dt}\left(\frac{\rho_\alpha}{\rho}h_\alpha\right) = \frac{dp_\alpha}{dt} - \nabla\cdot\mathbf{q}_\alpha + \hat{\boldsymbol{\tau}}_\alpha : (\nabla\mathbf{v}) + \rho_\alpha\mathbf{V}_\alpha\cdot\left(\mathbf{F}_\alpha - \frac{d\mathbf{v}}{dt}\right) - \sum_\beta R^{(e)}_{\alpha\beta} + W^{(e)}_\alpha. \tag{30}$$

Summing equations (29) over α, one obtains the equation of conservation of energy of the mixture

$$\rho\frac{du}{dt} = -\nabla\cdot\mathbf{q} - p\nabla\cdot\mathbf{v} + \hat{\boldsymbol{\tau}} : (\nabla\mathbf{v}) + \sum_\alpha \rho_\alpha\mathbf{V}_\alpha\cdot\mathbf{F}_\alpha + \sum_\alpha W^{(e)}_\alpha \tag{31}$$

or in terms of the enthalphy h of the mixture

$$\rho\frac{dh}{dt} = \frac{dp}{dt} - \nabla\cdot\mathbf{q} + \hat{\boldsymbol{\tau}} : (\nabla\mathbf{v}) + \sum_\alpha \rho_\alpha\mathbf{V}_\alpha\cdot\mathbf{F}_\alpha + \sum_\alpha W^{(e)}_\alpha, \tag{32}$$

where

$$h = \sum_\alpha \frac{\rho_\alpha}{\rho}h_\alpha = u + \frac{p}{\rho}. \tag{33}$$

5 Diffusion model with multiple temperatures

The terms on the lhs of equation (27) and the viscous stress terms and $\mathbf{W}^{(m)}_\alpha$ on the rhs are of the order of Knudsen number relative to $\mathbf{r}^{(m)}_{\alpha\beta}$ and the pressure gradient term and may be dropped. Then equations (27) assume the form of transport equations, relating the diffusion velocities and the diffusion forces, and coincide with the Stefan-Maxwell equations for multicomponent diffusion in multispecies gas mixtures in the conventional hydrodynamics theory, written in the first approximation; see [7] for details.

The contribution of the electrons to ρ and mass transport (equation (19) is negligible. Summing equations (29) over heavy particles (index h), one obtains the equation of conservation of energy of heavy particles

$$\rho\frac{du_h}{dt} = -\nabla\cdot\mathbf{q}_h - p_h\nabla\cdot\mathbf{v} + \hat{\boldsymbol{\tau}}_h : (\nabla\mathbf{v}) + \sum_{\alpha=h} \rho_\alpha\mathbf{V}_\alpha\cdot\mathbf{F}_\alpha + \sum_{\alpha=h} R^{(e)}_{ea} + \sum_{\alpha=h} W^{(e)}_\alpha, \tag{34}$$

where

$$u_h = \sum_{\alpha=h}\frac{\rho_\alpha}{\rho}u_\alpha, \quad \mathbf{q}_h = \sum_{\alpha=h}\mathbf{q}_\alpha, \quad p_h = \sum_{\alpha=h}p_\alpha, \quad \hat{\boldsymbol{\tau}}_h = \sum_{\alpha=h}\hat{\boldsymbol{\tau}}_\alpha. \tag{35}$$

Equation (34) coincides with equation (14.4-1) on p. 457 of [6]. Note that $p_h = \frac{2}{3}\rho u_h$.

Summing equations (30) over heavy particles, one obtains the equation of conservation of energy of heavy particles in terms of their enthalpy h_h

$$\rho\frac{dh_h}{dt} = \frac{dp_h}{dt} - \nabla\cdot\mathbf{q}_h + \hat{\boldsymbol{\tau}}_h : (\nabla\mathbf{v}) + \sum_{\alpha=h} \rho_\alpha\mathbf{V}_\alpha\cdot\mathbf{F}_\alpha + \sum_{\alpha=h} R^{(e)}_{ea} + \sum_{\alpha=h} W^{(e)}_\alpha, \tag{36}$$

where

$$h_h = \sum_{\alpha=h}\frac{\rho_\alpha}{\rho}h_\alpha = u_h + \frac{p_h}{\rho}. \tag{37}$$

Note that if the distribution functions of the heavy particles are maxwellian with the same temperature T_h, which is the usual assumption for thermal plasmas, then

$$p_h = n_h k T_h, \quad u_h = \frac{3}{2}\frac{n_h k T_h}{\rho}, \quad h_h = \frac{5}{2}\frac{n_h k T_h}{\rho}, \tag{38}$$

where $n_h = \sum_{\alpha=h} n_a$ is the number density of the heavy particles.

The equation of conservation of energy of the electrons is given by equation (29) for $\alpha = e$

$$\rho \frac{d}{dt}\left(\frac{\rho_e}{\rho} u_e\right) = -\nabla\cdot\mathbf{q}_e - p_e \nabla\cdot\mathbf{v} + \hat{\boldsymbol{\tau}}_e : (\nabla\mathbf{v}) + \rho_e \mathbf{V}_e \cdot \left(\mathbf{F}_e - \frac{d\mathbf{v}}{dt}\right) - \sum_{\alpha=h} R_{e\alpha}^{(e)} + W_e^{(e)}. \tag{39}$$

Note that $p_e = \frac{2}{3}\rho_e u_e$. The viscous stress term on the rhs is small due to the smallness of the electron mass and may be dropped, as well as the term proportional to $d\mathbf{v}/dt$.

Equation (39) conforms to equation (5.4) on p. 189 of [8], to the accuracy of the viscous stress term and the term proportional to $d\mathbf{v}/dt$ (which are small, as indicated above). On the other hand, equation (39) does not conform to equation (14.4-2) on p. 457 of [6], since the material derivative term on the lhs of the latter equation is written in the form $\rho_e\, du_e/dt$.

Eq. (39) may be rewritten in terms of the electron enthalphy h_e

$$\rho \frac{d}{dt}\left(\frac{\rho_e}{\rho} h_e\right) = \frac{dp_e}{dt} - \nabla\cdot\mathbf{q}_e + \hat{\boldsymbol{\tau}}_e : (\nabla\mathbf{v}) + \rho_e \mathbf{V}_e \cdot \left(\mathbf{F}_e - \frac{d\mathbf{v}}{dt}\right) - \sum_{\alpha=h} R_{e\alpha}^{(e)} + W_e^{(e)}. \tag{40}$$

Note that if the distribution function of the electrons is maxwellian with the temperature T_e, which is the usual assumption for thermal plasmas, then

$$p_e = n_e k T_e, \quad u_e = \frac{3}{2}\frac{k T_e}{m_e}, \quad h_e = \frac{5}{2}\frac{k T_e}{m_e}. \tag{41}$$

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